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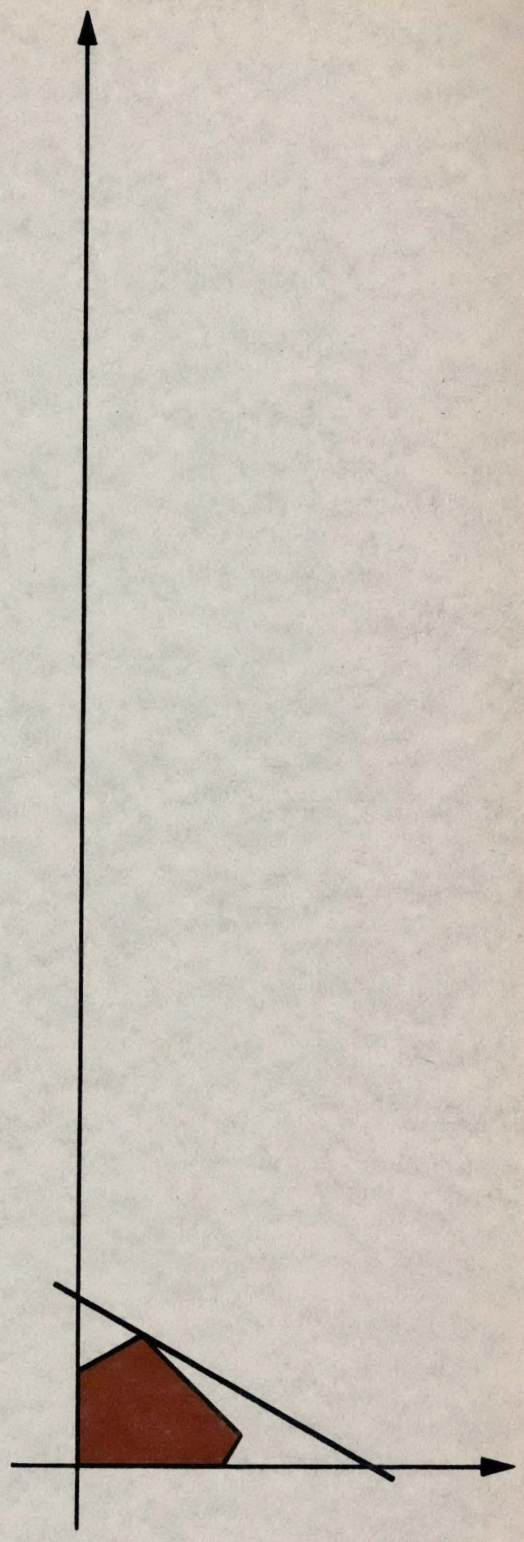

OPERATIONS RESEARCH CENTER

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A PRELIMINARY MODEL FOR INITIAL ATTACK

I. Introduction

It is generally agreed among those concerned with forest fire control that a strong initial attack will prevent most fires from causing substantial damages, both in values burned and in suppression costs. The question of "how soon" and "how much" manpower and equipment should be committed to the initial attack does not, as yet, admit of a clear-cut answer. The response "everything you have got as soon as possible" is not satisfactory since it begs the fundamental questions of how much should "you have" available for dispatch and what are the economic considerations involved in "as soon as possible." Before these questions can properly be analyzed, a realistic model of fire growth versus initial attack is needed for proper pre-suppression planning. This paper is a step in that direction.

The simple deterministic model to be presented here does not pretend to portray exactly the course of every wildland fire, nor to predict precisely the effect of various allocations and tactics of using suppression forces. Rather, it attempts to describe in a "broad-brush" manner the general pattern of a fire, and a probable mechanism for its suppression. The results are rather general indications as to the economical sizes of suppression forces one should use for fires of various sizes and growth potentials, and, perhaps even more importantly, the effects of the various physical and economical parameters and groups of parameters upon the final costs of suppression and burn. In this way we shall get an idea of the underlying structure of the initial attack problem. Comparison of the results thus obtained against actual experiences and intuition will, we hope, suggest ways to improve the model, but will not, we suspect, directly contradict it.

II. The Fire Growth Model

The general pattern of fire growth can be pictured as in Figure 1. At time T_1 ignition of the fire takes place, but it is not detected until some time later, at T_D. Because of the delay in mobilizing and transporting men and equipment, the actual attack on the fire does not usually begin until an even later time, T_A. The size of the fire, which has been growing steadily or even accelerating up to this time, still continues to grow after the initial attack; but its rate of growth is slowly decreased until, at time T_C, the fire is brought under control with a total area of burn y_C. From T_C to T_M, mop-up of the burned area takes place, until final extinction occurs at T_F.

Thus, the ultimate area (and cost) of a fire is seen to depend upon three critical time intervals. By spending more funds upon lookouts, aerial surveys, and other surveillance devices, the detection interval, τ_D, could be reduced. By investing in more rapid means of transport, such as helicopters, and by the staffing of seasonal or year-round ready crews, the attack interval, τ_A, may be shortened. Finally, by sending more suppression forces and equipment, the control interval, τ_C, may usually be decreased. Mop-up usually proceeds at a leisurely pace, and thoroughness, rather than time elapsed, is a more important criterion for this operation.

In this report we shall concentrate our attentions upon the time interval from T_A to T_C, the control interval, and will investigate the effect of allocating varying amounts of suppression forces on the fire. If this decision is to be based solely upon the parameters of the fire reported at the time of detection, T_D, then two physical "laws" of behavior are needed:

(1) A description of fire behavior during the interval from detection until attack.

(2) A description of the effect of a suppression force of a given size against a fire of known physical characteristics.

Unfortunately, it is in these two areas that fire control theory is lacking in results. There is, at present, no statisfactory macro-physical model of fire spread in existence; some fire-growth predictors currently available are useful for local extrapolations when fire conditions are changing. There are various guidelines available on the number of forces required to suppress a fire of a given size; but, as near as we can tell, no studies have been made on the relationship between time until suppression for varying levels of suppression activity.

Not having an available model of fire growth, we shall be forced to assume one which is reasonable, showing the correct behavior as various parameters are changed, or as various limiting situations are tested. We shall have more to say later about the philosophy and trustworthiness of operating with such simple models.

The basic model of this paper assumes that, in the absence of suppression forces, the growth of fire consists of two deterministic components: a linear growth with time, and an accelerating component. Thus, if $t = 0$ represents time measured from time of detection, the growth of fire may be represented by:

$$\frac{dy}{dt} = G_D + Ht \qquad (1)$$

$$y(0) = y_D \quad .$$

Here y_D is the observed area of burn at time T_D, G_D is the observed rate of growth (acres/hour, etc.), and H is acceleration component of growth (acres/hour^2). By an appropriate choice of parameters, one can choose a fire which is increasing its area in a linear fashion, or "simulate" one which is doubling in size every 24 hours, etc.

At the time of attack, T_A, the fire has grown to a size

$$y_A = y(\tau_A) = y_D + G_D\tau_A + \tfrac{1}{2}H(\tau_A)^2 \tag{2}$$

and is growing at a rate

$$G_A = y'(\tau_A) = G_D + H\tau_A \tag{3}$$

which is faster than that observed at T_D due to the acceleration of the fire.

For example, if a fire when detected were observed to have the parameters

$$G_D = 3.5 \text{ acres/ hour} \quad \text{and} \quad H = 3.0 \text{ acres/hour}^2 ,$$

then in the next half hour, the fire would consume an additional 2 1/8 acres, and would then be burning at a rate of spread 5.0 acres/hour. In the next half hour after that the fire, if unchecked, will burn 2 7/8 more acres more, and be growing at a rate of 6.5 acres/hour.

At time T_A we have the possibility of sending in suppression forces. The effect of such forces in our model will be described as growth-rate decreasing, or "deaccelerating" forces. Thus, from the high point of growth rate at T_A, the forces slowly remove the momentum of the fire, until its growth rate is zero; at this point, the fire is under control. In mathematical terms, the growth rate after time T_A is described by:

$$\begin{aligned} \frac{dy}{dt} &= G_D + Ht - E(t - \tau_A)x \qquad t \geq \tau_A \\ &= G_A - (Ex - H)(t - \tau_A) \end{aligned} \tag{4}$$

That is, after T_A the fire has an effective deceleration $Ex - H$. Here x is a decision variable which represents the number of supression forces

sent in against the fire, and E is an "efficiency" factor which represents the effect of different types of forces, training, equipment available, etc. measured in area/man-time[2]. This factor E will undoubtedly be a function of y_A, the fire size at time of initial attack. It will depend on the number of chains (length), if not of the fire perimeter, at least of the fire front. It what follows we will use E as a constant, and refer only to fires that are "small" at time T_A, say, less than one acre.

We may think of this deceleration force in the following way: If the fire is spreading at a linear rate, then a man is capable of extinguishing by various means a certain number of linear feet of fire per unit time period. When this force is greater than the growth spread, we have "effective deceleration."

In what follows we will refer to a "man" as a unit of fighting force, but of course, it could be any size fighting unit, a bulldozer, "standard crew," etc. The effect of these "men" is assumed to be linear, that is, we obtain twice as much effect with twice as many men, and it is assumed to be constant throughout the life of the fire, that is, there are no "fatigue" effects.

From Equation (4), we can solve directly for τ_C, the interval of time until control. We have:

$$\tau_C = \frac{G_A}{(Ex - H)} = \frac{(G_D + H\tau_A)}{(Ex - H)} \quad . \tag{5}$$

By integrating Equation (4), we can find the additional area burned during the control interval:

$$y_C - y_A = \int_{\tau_A}^{\tau_C+\tau_A} \frac{dy}{dt}\, dt = \int_0^{\tau_C} \left\{ G_A - (Ex - H)u \right\} du = G_A\tau_C - \tfrac{1}{2}(Ex - H)(\tau_C)^2$$

and substituting (5), we have:

$$(6) \qquad y_C - y_A = \frac{(G_A)^2}{2(Ex - H)} \quad .$$

In Figure 2 is shown the type of fire growth pattern which is predicted by our model with G_D = 3.5 acres/hour, H = 3 acres/hour^2, τ_A = 0.5 hours, E = 0.1 acres/hour^2-man, and 44, 54, and 64 men. It is seen that the growth is composed of two parabolic portions, one where the fire spreads and accelerates, unchecked during the attack interval, and a decreasing parabola where the fire is being slowed down by the suppression forces. By direct calculation, we see that the control interval, and (controllable) area of burn are:

	44 men	54 men	64 men	
Control Interval, τ_C	3.57	2.08	1.77	hours
(Controllable) Area of Burn, $y_C - y_A$	8.93	5.21	3.68	acres

Thus, by putting more suppression forces into the fire, both the time to control and the total acreage burned are decreased; this is certainly in agreement with actual experience.

We may investigate this relationship in more detail by examining the form of (5) and (6). Notice that the fire cannot be brought under control at all unless Ex is greater than H . This is the same thing as saying that there is some minimal suppression force

(7) $$x_0 = \frac{H}{E}$$

which is required. From (4), it can be seen that this minimal force is just the deceleration factor which "kills off" the acceleration of the fire, and straightens out the growth curve into a straight line; we might say that these men are keeping the fire "under control," although they are certainly not preventing it from spreading – additional manpower is required to do that.

If we use the variable z to denote the number of forces above and beyond x_0, i.e.,

(8) $$z = x - x_0 \quad ,$$

then we see that the control interval becomes:

(9) $$\tau_C = \frac{G_A}{Ez}$$

(10) $$y_C - y_A = \frac{(G_A)^2}{2Ez} \quad .$$

Both of these quantities are inversely proportional to the number of additional men employed; thus both curves have the form shown in Figure 3. This result appears to be quite realistic. It states that there is some minimal number of forces which is needed to make any progress against the fire at all; beyond that point there is an inverse relationship between the number of forces and the time and area of burn.

In actual practice, forest personnel may prefer to estimate the approximate shape of a curve like Figure 3, rather than estimate the individual parameters G_A, H, E, etc. Our purpose has been to show that a simple operational model could lead to a realistic description of system behavior.

The influence of the effectiveness of parameter E in (5) and (6) is not too surprising, since E multiplies the manpower to get fire deceleration. However, the growth rate of fire at attack time, G_A, enters in two different ways. Equation (5) states that the control interval is directly proportional to G_A; in (6), however, we find that the ultimate area of burn will be proportional to the square of the growth rate at attack (for the same number of men)! Stated in other words, if we have two fires, one of which is spreading twice as fast as the other, then we need twice as many men (above and beyond x_0) to put out the fire in the same time; but if our object is to confine the fire to the same area, then we need four times as many additional men!

G_A also contains the effect of attack time through the relationship $G_A = G_D + H\tau_A$. We shall investigate this dependence in a later section. Let us now consider the economics of sending a certain size suppression force.

III. The Economics of Suppression

The size of the suppression force to be sent to a fire depends upon a balance between several economic factors. Thus, it is possible to send a large number of men, and thus drastically limit the area of burn; this policy will incur a lot of transportation expense. On the other hand, one may send just a few men to fight the fire incurring a large loss in acreage burned and property destroyed. The proper balance of these economic factors will be the subject of this section.

We suppose that there are four types of cost which can be incurred in suppressing a given fire:

1. A fixed cost, associated with setting the fire suppression organization in action, and denoted by C_F (in dollars).
2. A cost which is proportional to the number of forces employed in suppressing the fire. This cost is primarily one of transportation and the other "one-shot" logistic support costs, and will be represented by C_S (in dollars/man).

3. A cost which is proportional to the total manpower-hours spent in suppressing the fire. The unit cost of this portion of the operation, C_X (dollars/man-hour) reflects the wages of the suppression forces as well as the various expendable and continuing support items.

4. A cost which is proportional to the area of total burn, represented by C_B (in dollars/acre).

Various other costs with these "dimensions" may be incorporated in the parameters just described. For instance, the cost of mop-up operations is probably proportional to the area burned, but has nothing to do with the size of the force which suppressed the fire; this cost would then be included in C_B.

Since bookkeeping records all assume (linear) unit costs of operations,* there is probably no easy way to describe continuously nonlinear costs, even if they should occur. A more important omission of this model, which can be corrected, is the discontinuous nonlinear costs which are associated with varying the number of suppression forces. Thus, the cost of sending one man more than a standard size crew clearly introduces some nonproportional costs!

We shall neglect this indivisibility problem for the moment, and assume that all of the costs incurred are linear and additive. We may then set up the total cost equation as:

$$\text{Total Cost} = C_F + C_S(\text{units of force}) + C_X(\text{number of units times hours needed}) + C_B(\text{area burned})$$

or

$$C = C_F + C_S x + C_X \tau_C x + C_B y_C$$

*With the possible exception of watershed valuations on the Southern California forests.

or

$$C = C_F + C_S x + C_X \frac{G_A}{Ex - H} x + C_B \left[y_A + \frac{G_A^2}{2(Ex - H)} \right] \tag{11}$$

where y_A is given by Equation (2).

We note that C_S will depend, to a large extent, on the method of transport used (foot vs. truck vs. helicopter, etc.), and this in turn will influence the interval between detection and attack, τ_A. Later we will want to examine the effect on the total costs of varying C_S, and hence τ_A; for the present we merely point out that there is a functional relationship between the two quantities.

Using the relationship (8) to eliminate the fixed base of suppression force which must be sent anyway, we find that the total cost as a function of the additional forces sent may be written as:

$$C(z) = C_0 + C_S z + \frac{\left(\frac{C_X G_A H}{E^2} + \frac{C_B G_A^2}{2E} \right)}{z} \tag{12}$$

where

$$C_0 = \left[C_F + \frac{C_S H}{E} + \frac{C_X G_A}{E} + C_B y_A \right] \tag{13}$$

is a constant cost, independent of the number of additional forces sent.

We may readily identify the various portions of (12) in their effect upon the total cost if various actions are taken. For instance, if one sends in a large number of suppression forces, z will be large, and the second term in (12) will be dominant; that is, our cost will be largely due to travel

and other one shot items which are proportional to the size of the force which is employed. On the other hand, if few men are sent in, then the last term in (12) becomes dominant, and two factors are contributing to our cost — the cost of the burned area, and the man power-hour cost. It is interesting to note that manpower-hour costs are higher the fewer additional men are sent; this is because the time that the total forces must spend extinguishing the fire is determined disproportionately by the few extra men, thus increasing total manpower-hours.

If there are high cost penalities associated with too many or too few men, this suggests that there may be some intermediate point of operation which reduces the total cost to a minimum value — that is, it may be possible to find an optimal level of attack for the simple model and the assumptions just outlined. To find this optimal level of suppression, we take a short mathematical detour to obtain some results in the mathematical theory of optimization which apply to the special form of (12).

THEOREM: If $C(z) = \alpha z + B/z + \gamma$, then $C^* = C(z^*) = \underset{z=z^*}{\text{minimum}}\ C(z)$ is attained at the point $z^* = \sqrt{\beta/\alpha}$ with $C^* = 2\sqrt{\alpha\beta} + \gamma$. Furthermore, $C''(z^*) = 2\alpha\sqrt{\alpha/\beta}$, and $C'''(z^*) = -6\alpha^2/\beta$.

PROOF: The necessary condition for a stationary point is that dC/dz be zero, i.e., $\alpha - \beta/z^2 = 0$. This occurs at the points $z = \pm\sqrt{\beta/\alpha}$. The second derivative is $d^2C/dz^2 = 2\beta/z^3$, and a sufficient condition that the stationary point just found is a minimum in that $C''(z) > 0$. This occurs at $z^* = +\sqrt{\beta/\alpha}$, with $C''(z^*) = 2\alpha\sqrt{\alpha/\beta}$, and $C'''(z^*) = -6/z^4 = -6\alpha^2/\beta$. Upon substitution, we find that the two variable components of the cost function are equal at optimality, with $C^* = 2\sqrt{\alpha\beta} + \gamma$.

With these results, we can find the optimal initial attack force from Equation (12) after making a correspondence between the various constants.

We find that the optimal number of additional forces to be sent to the fire is:

$$z^* = \sqrt{\frac{\left(\frac{C_X G_A H}{E^2} + \frac{C_B G_A^2}{2E}\right)}{C_S}} = \sqrt{\frac{C_B G_A^2 E + 2C_X G_A H}{2E^2 C_S}} \quad . \tag{14}$$

In many operations the cost of manpower-hours is small compared to the cost of the burned area. By defining ϵ_X as the ratio:

$$\epsilon_X = \frac{C_X H}{C_B G_A E} = \frac{C_X x_0}{C_B G_A} \tag{15}$$

(i.e., the ratio of cost rate for the basic minimal force to the cost rate of burning wildland at time of attack), we can rearrange (14) as:

$$z^* = G_A \sqrt{\frac{C_B}{2C_S E}}\ \sqrt{1 + 2\epsilon_X} \tag{16}$$

where if $\epsilon_X \ll 1.0$, $\sqrt{1 + 2\epsilon_X} \approx 1 + \epsilon_X$, a small correction term. The optimal total suppression force is, of course,

$$x^* = z^* + x_0 = z^* + \left(\frac{H}{E}\right) \quad . \tag{17}$$

By substitution, we find that the minimal cost incurred by sending the optimal number of suppression forces is:

(18) $$C^* = C_O + 2G_A \sqrt{\frac{C_S C_B}{2E}} \sqrt{1 + 2\epsilon_X} \quad .$$

Later, we will examine the effect of sending a nonoptimal number of men on an initial attack, both too few and too many. For this purpose, we also record at this point the second and third derivatives of $C(z)$ at the optimal point $z = z^*$.

(19) $$C''(z^*) = \frac{2}{G_A} \sqrt{\frac{2EC_S^2}{C_B}} \frac{1}{\sqrt{1 + 2\epsilon_X}}$$

(20) $$C'''(z^*) = - \frac{12EC_S^2}{C_B G_A^2} \frac{1}{(1 + 2\epsilon_X)}$$

At this point, a numerical example may be helpful. Let us reconsider the same fire which served as a previous example, where

$G_D = 3.5$ acres/hour $\qquad H = 3.0$ acres/hour^2

$\tau_A = 0.5$ hours $\qquad E = 0.1$ acres/hour^2-man

and let us assume that $y_D = 1.0$ acre. Furthermore, we assume that the various cost parameters for this particular fire are:

$C_F = 1{,}000$ dollars $\qquad C_S = 50$ dollars/man

$C_X = 3$ dollars/man-hour $\qquad C_B = 200$ dollars/acre. (including mop-up)

From the previous calculations, we see that at the time of attack, a total of $y_A = 1 + 2\ 1/8 = 3\ 1/8$ acres will be burning, with an instantaneous growth

rate of $G_A = 3.5 + 3/2 = 5.0$ acres/hour. The required minimal number of forces is $x_0 = 3/0.1 = 30$ men. From (12), we calculate that the fixed cost of operation will be:

$$C_0 = 1{,}000 + (50 \times 30) + \left(\frac{3 \times 5}{0.1}\right) + \left(\frac{200 \times 25}{8}\right) = \$3{,}275 \quad .$$

Also, $C_B G_A^2 / 2E = 25{,}000$ dollar-men ,

$C_X G_A H / E^2 = 4{,}500$ dollar-men ,

$\epsilon_X = 0.09$.

The total cost of fire suppression is:

$$C(z) = 3{,}275 + 50z + 29{,}500/z \text{ dollars} \quad .$$

The cost function is plotted in Figure 4 vs. z , and the individual components are also shown and labelled as individual curves. From (16) we calculate:

$$z^* = 5\sqrt{20}\ \sqrt{1.18} = 24.3 \text{ men}$$

are required above and beyond x_0 , making a total of about $x^* = 54$ men to be sent to the fire.

When this is done, 1215 additional dollars are spend for transportation and other one-time expenses, about 1,030 dollars are lost in burned acreage past the time of attack, and about 185 dollars are spent in manpower-hour dollars, making a total (minimal) variable cost of 2,430 dollars, for a grand total minimum cost of $C^* = 5{,}705$ dollars. After breaking down the fixed cost component C_0 , we find that the total cost was spent as follows:

Fixed Cost of Operation	$1,000
Transportation-type Expenses	2,715
Manpower-hour Expenses	335
Total Burned Acreage Cost	1,655
Approx, Total Cost of Fire	$5,705

The time to control the fire is about $\tau_A = 2.06$ hours, with a total area of burn of about $y_C = 8.28$ acres. The progress of the fire is shown as the middle curve in Figure 2. (The slight discrepancies with the table above are due to the fact that exactly 54 men were used in that calculation.)

This example is intended as a representative one only, and should not be interpreted as "realistic," or "unrealistic."

We may want to consider introducing another element into the total cost structure, say C_T, in units of dollars/time. We may think of this as a "mobilization" cost, "alert" cost, or some other cost that is proportional only to the length of time the fire is burning out of control. This would then give us expressions to take care of costs for all units in the model; dollars, dollars/acre, dollars/man, dollars/man-hour, and now dollars/hour.

In the total cost expression we would have another term $C_T t$ where t equals elapsed time in which the fire is out of control = $\tau_C = G_A/Ez$. We may combine $C_T G_A/Ez$ with $C_B G_A^2/2Ez$ and thus obtain a new C_B, say C_B', which equals $C_B + 2C_T/G_A$. We may then substitute this into equations (12) and (13) and solve in a similar way for z^*, C^*, and τ_A^*.

IV. Sensitivity Analysis

Now let us examine what happens as we move away from the optimal number of suppression forces just determined in the previous section. For small changes, we can expand the resulting cost in a Taylor's Series:

$$(22) \qquad C(z^* + u) = C(z^*) + C''(z^*)\frac{u^2}{2} + C'''(z^*)\frac{u^3}{6} + \ldots$$

The linear term is missing since the series is expanded about the minimum cost point $z = z^*$.

We see that to a first approximation the change in cost for a nonoptimal choice of z gives a parabolic change in cost, while for larger deviations, a cubic term must be added to represent the skewness of Figure 4. The values of the second and third derivatives for our problem are given by Equations (19) and (20).

For the example of the last section,

$$C''(z^*) = 2(2.05847) \text{ dollars/man}^2 \qquad C'''(z^*) = -6(0.08475) \text{ dollars/man}^3$$

If $u = \pm 10$, that is we send 10 too many (or too few) men to the fire, we calculate directly that:

$$C(34.3) = 5{,}705 + 2.06(100) - 0.085(1{,}000)$$
$$= 5{,}705 + 206 - 85 = \$5{,}826$$

and

$$C(14.3) = 5{,}705 + 206 + 85 = 5{,}996 \text{ dollars} \quad .$$

Thus, there is not too much loss in either case, for our example; on the other hand, if we take $u = \pm 20$ men, that is we send 20 too many (or too few) men:

$$C(44.3) = 5{,}705 + 823 - 678 = \$5{,}850$$
$$C(4.3) = 5{,}705 + 823 + 678 = \$7{,}206$$

The extreme difference in this case reflects that the cost curve, although fairly flat in our example, is highly skewed, as the cost for sending too few men is considerably higher than the cost of sending too many. This

implies that if some error is to be made in estimating parameters of the fire, it is better to "overkill" the fire and incur the extra transportation and other "one-shot" costs, than it is to send too few men, and lose area to the fire.

We may also investigate the relationship between the optimal number of forces and the resulting cost for changes in the parameters of the problem, under the assumption that we know what the changes will be, and plan our strategy accordingly. Figures 5 through 16 illustrate $C(z^*)$ and z^* vs. G_D , H , E , C_S , C_B , and τ_A for the sample problem. We see that in some cases there is a wide change in strategy employed, while in other cases the sensitivity is much smaller. A knowledge of which parameters affect the solution strongly will give an indication of which parameters must be known accurately, and which need only be roughly determined.

For small changes in the parameters, we can assume that the curves in Figures 5 through 16 are approximately straight lines near the values of the parameter which was used to calculate the original z^* , and we can get the slope of this tangent approximation by differential calculus. For example, if G_D is to vary,

$$\frac{\partial C}{\partial G_D} = \frac{\partial G_A}{\partial G_D} \cdot \frac{\partial C}{\partial G_A} + \frac{\partial y_A}{\partial G_D} \cdot \frac{\partial C}{\partial y_A}$$

since no other parameters depend on G_D , and

$$\frac{dG_A}{dG_D} = 1 \qquad \frac{\partial y_A}{\partial G_D} = \tau_A \qquad \frac{\partial C}{\partial G_A} = \frac{C_X}{E} + \frac{\dfrac{C_B G_A}{E} + \dfrac{C_X H}{E^2}}{z}$$

or, $\partial C/\partial y_A = C_B$

(23)
$$\frac{\partial C}{\partial G_D} = \frac{C_X}{E} + \frac{\frac{C_B G_A}{E} + \frac{C_X H}{E^2}}{z^*} + C_B \tau_A \quad .$$

At the point $z = z^*$, we calculate for our numerical example,

$$\frac{\partial C}{\partial G_D} = 30 + \frac{10{,}000 + 900}{24.3} + 100 = 579 \ \frac{\text{dollar-hours}}{\text{acre}} \quad .$$

This means that an increase of 1 acre/hour in G_D should cause an increase in cost of about $579. Indeed, if we resolve the numerical example using $G_D = 4.5$ acres/hr. letting z^* change, we find that $y_A = 3.625$ acres, $G_A = 6$ acres/hr., $\epsilon_X = 0.075$, $z^* = 28.8$, $C^* = \$6{,}283$, a "time"change of $578, which is close enough.

If only the change in manpower is desired, then from (16) we can find:

$$\frac{\partial z^*}{\partial G_D} = \frac{\partial z^*}{\partial G_A}\frac{\partial G_A}{\partial G_D} + \frac{\partial z^*}{\partial \epsilon_X}\frac{\partial \epsilon_X}{\partial G_D} \qquad \text{with} \qquad \frac{\partial \epsilon_X}{\partial G_D} = \frac{\partial \epsilon_X}{\partial G_A}\frac{\partial G_A}{\partial G_D}$$

we have

$$\frac{\partial G_A}{\partial G_D} = 1 \qquad \frac{\partial \epsilon_X}{\partial G_D} = -\frac{C_X H}{C_B E G_A^2}$$

$$\frac{\partial z^*}{\partial G_A} = \sqrt{\frac{C_B}{2C_S E}}\sqrt{1 + 2\epsilon_X} \qquad \frac{\partial z^*}{\partial \epsilon_X} = G_A\sqrt{\frac{C_B}{2C_S E}}\frac{1}{\sqrt{1 + 2\epsilon_X}}$$

or

(24)
$$\frac{\partial z^*}{\partial G_D} = \frac{z^*}{G_A} - \frac{\epsilon_X z^*}{G_A(1 + 2\epsilon_X)} = \frac{z^*}{G_A}\left(\frac{1 + \epsilon_X}{1 + 2\epsilon_X}\right)$$

where the old values of G_A , z^* , ϵ_X are used. For our example, we find:

$$\frac{\partial z^*}{\partial G_D} = 4.49 \ \frac{\text{man-hours}}{\text{acre}} \quad .$$

That is, if the initial growth rate increased by 1 acre/hr., we should expect optimally, to add 4.5 men. From our previous calculations we see that we do indeed add 28.8 - 24.3 = 4.5 men, so that the linear approximation is, in this case, a very good one.

Other dependencies of C^* and z^* on the parameters can be explored in a like manner.

V. Influence of Parameters

Before considering some advanced topics, it is desirable to investigate the effect of the parameters in the formulas for optimal level of suppression and resulting minimal cost. For convenience, we repeat these formulas.

$$x^* = \frac{H}{E} + G_A \sqrt{\frac{C_B}{2C_S E}} \sqrt{1 + 2\epsilon_X}$$

$$C^* = \left[C_F + \frac{C_S H}{E} + \frac{C_X G_A}{E} + C_B y_A \right] + 2G_A \sqrt{\frac{C_S C_B}{2E}} \sqrt{1 + 2\epsilon_X}$$

with

$$\epsilon_X = \frac{C_X H}{C_B G_A E} \quad ,$$

$$G_A = G_D + H\tau_A \quad ,$$

and

$$y_A = y_D + G_D \tau_A + \frac{H(\tau_A)^2}{2} \quad .$$

The first feature of interest is that, except for the factor ϵ_X which is usually not important, both x^* and C^* depend linearly upon G_A – that is, upon G_D and τ_A. Thus, if the growth rate of the fire (or the interval of attack) increases, both the number of forces and the resulting cost increase proportionately; if both G_D and τ_A increase, there is a faster than linear increase in the term $C_B y_A$ – the cost of burned area at time of attack.

The acceleration of the fire, H, affects primarily the minimal required force x_0 required for fixed G_A; however if we compare problems with the same growth rate at detection, G_D, we obtain a direct increase in the variable number to be sent as well. Cost is linear in H.

Use of forces with increased efficiency, E, in general decreases the values of x^* and C^*. The dependence is of mixed type in E^{-1} and $E^{-0.5}$. Notice that there is no single "trade-off" between forces of differing efficiency and costs which permits a simple comparison between different "technologies" of fighting fires.

The behavior of the optimal values x^* and C^* with respect to the costs is quite instructive. First, we note that by definition C_F will be the same regardless of the fire parameters or the attack decisions made. C_S enters in a more complicated way. As its value increases, we send less men z^* to the fire, but we lose more area to the fire. We must send at least H/E men so we must incur $C_S H/E$ dollars of cost, but beyond that there is a trade-off between costs of sending additional men, hence putting the fire out more quickly, and sending fewer men, hence incurring a greater cost of area burned. As C_B increases we note that we increase x^* roughly proportional to the square root of C_B. If C_B is large, it may be worthwhile to attempt to reduce τ_A and thus y_A to keep C^* from increasing. This would also decrease G_A, and hence have a decreasing effect on another term in C^*, $2G_A\sqrt{C_S C_B/2E}\sqrt{1 + 2\epsilon_X}$.

We note that C_X has only a small effect on x^*, entering only through the term $\sqrt{1 + 2\epsilon_X}$, but its effect on C^* is more direct, being directly proportional to G_A/E. This fact would also suggest the advisability of decreasing τ_A in order to decrease y_A and G_A as above. A rather interesting result occurs in the solution for the interval of control, τ_C, if an optimal policy is followed. We have:

$$\tau_C^* = \frac{G_A}{Ez^*} = \frac{2C_S}{C_B E} \frac{1}{\sqrt{1 + 2\epsilon_X}} \quad . \tag{25}$$

Except for possible variations in ϵ_X and a possible functional relationship between E and G_A, τ^* does not depend on G_A – that is, the optimal policy keeps (almost) all fires burning for the same length of time after the attack of suppression forces!

VI. Problems of Risk

The values of the parameters will not be known with certainty and they will change over time, but we must decide on a course of action x^* to take in advance. For instance, assume G_A is not known with certainty but is given by some probability distribution. What is the best x^* to choose a priori? For this purpose, it will be reasonable to take expected values of the cost over many trials (fires).

We may write the general cost equation as a function of z and G_A as:

$$C(z,G_A) = C_B y_A + C_F + C_S \frac{H}{E} + \frac{C_X G_A}{E} + C_S z + \frac{\left(\frac{C_X G_A H}{E^2} + \frac{C_B G_A}{2E}\right)}{z} \quad \text{(from Eq. (12))}$$

Taking expected values, we obtain:

$$(26) \qquad E\left[C(z)\right] = C_B y_A + C_F + C_S \frac{H}{E} + \frac{C_X \hat{G}_A}{E} + C_S z + \frac{\left(\frac{C_X \hat{G}_A H}{E^2} + \frac{C_B \hat{G}_A^2}{2E}\right)}{z}$$

Now let

$$(27) \qquad k^2 = \frac{\text{variance of } G_A}{(\text{mean of } G_A)^2} = \frac{\sigma^2}{u^2}$$

Since σ^2 = "expected value squared minus square of expected value" equals $\widehat{(G_A^2)} - (\hat{G}_A)^2$, $\widehat{G_A^2} = \sigma^2 + (\hat{G}_A)^2$. Now, from (27) $\sigma^2 = k^2(\hat{G}_A)^2$ and $\widehat{G_A^2} = k^2(\hat{G}_A)^2 + (\hat{G}_A)^2 = (k^2 + 1)(\hat{G}_A)^2$. Substituting this into (26)

$$E\left[c(z)\right] = C_B y_A + C_F + C_S \frac{H}{E} + \frac{C_X \hat{G}_A}{E} + C_S z + \frac{\frac{C_X \hat{G}_A H}{E^2} + \frac{C_B (k^2 + 1)(\hat{G}_A)^2}{2E}}{z}$$

and thus

$$(28) \qquad z^* = \sqrt{\frac{\frac{C_X \hat{G}_A H}{E^2} + \frac{C_B (k^2 + 1)(\hat{G}_A)^2}{2E}}{C_S}}$$

$$= \sqrt{\frac{C_B (k^2 + 1)(\hat{G}_A)^2 E + 2C_C \hat{G}_A H}{C_S}}$$

$$= \sqrt{k^2 + 1}\,(\hat{G}_A) \sqrt{\frac{C_B}{2C_S E}} \sqrt{1 + 2\hat{\epsilon}_X}$$

where $\hat{\epsilon}_X$ is now $C_X x_O / C_B \hat{G}_A$, which we note is the same answer as obtained in (16)

except that we use the expected value of G_A and multiply the result by $\sqrt{k^2 + 1}$! Note that if $\sigma^2 = 0$, that is, we know G_A with certainty, then $k^2 = 0$, and we obtain (16) as before. Note also that if there is any uncertainty in G_A , k^2 will be greater than 0 and $\sqrt{k^2 + 1}$ will be greater than 1 . This means that z^* will always be increased if uncertainty is present and we should always send more men than in the determinate case.

What will k^2 be for some distributions of interest?

Consider (i) uniform distribution over some interval
(ii) normal
(iii) "worst possible" – spread, given just an upper and lower limit, with probabilities of 1/2 concentrated at both limits.

We may picture these three cases as in Figure 17. For (i)

$$\mu = \frac{A + B}{2} \qquad \sigma^2 = \frac{(B - A)^2}{12}$$

$$\therefore \quad k^2 = \frac{\sigma^2}{\mu^2} = \frac{(B - A)^2}{12} \frac{4}{(A + B)^2} = \frac{(B - A)^2}{3(A + B)^2} = \frac{1}{12} \left[\frac{\text{range}}{\text{mean}}\right]^2$$

For (ii) assume range $= B - A = N\sigma$, $\therefore$ $\sigma = (B - A)/N$ and $\sigma^2 = (B - A)^2/N^2$

$$\mu = \frac{A + B}{2}$$

$$\therefore \quad k^2 = \frac{\sigma^2}{\mu^2} = \frac{\frac{(B - A)^2}{N^2}}{\frac{(A + B)^2}{4}} = \frac{4}{N^2} \frac{(B - A)^2}{(A + B)^2} = \frac{1}{N^2} \left[\frac{\text{range}}{\text{mean}}\right]^2$$

For instance, for "2σ limits" (range covers 95.46%/o of cases), $N = 4$. For (iii)

$$\mu = \frac{A + B}{2}$$

$$\sigma^2 = E(G_A^2) - \left[E(G_A)\right]^2 = \frac{1}{2}\left[A^2 + B^2\right] - \frac{(A+B)^2}{4} = \frac{(B-A)^2}{4}$$

$$k^2 = \frac{\sigma^2}{\mu^2} = \frac{(B-A)^2}{(B+A)^2} = \frac{1}{4}\left[\frac{\text{range}}{\text{mean}}\right]^2$$

Now let us examine what happens to z^* under case (i) above, the uniform distribution. Using our previous numerical example, but considering that G_A is uniformly distributed between 3.5 and 6.5, so that range = 3 and $E(G_A) = 5.0$, we find that we may solve (28) to get:

$$k^2 = \frac{9}{12.25} = .03 \quad , \qquad \sqrt{k^2 + 1} = 1.015 \quad .$$

or z^* 24.3 x 1.015 = 24.67 . That is, under these conditions, approximately 1/3 more men should be sent for an optimal solution if G_A is uniformly distributed between 3.5 and 6.5. Similar results could easily be obtained for the other cases above or for other distributions that may be of interest. In more "practical" applications, the effect of uncertainty will probably be greater.

VII. Reinforcement Policies

This very important problem deserves fuller treatment than will be given here, but the following analysis will show the sort of results that can be obtained from this model.

Assume that $G_A = G_D + HT_A$ was not properly forecast, and it is not known until time T_A that too few men were sent for an optimal solution. Therefore, reinforcements cannot arrive at the fire until at least time $T_A + \tau$ where τ is the travel time to the fire. Further assume that at least $x_0 = H/E$ men were sent in at first, plus $z_1 < z^*$ more. (If H was not forecast properly, define H' as the true acceleration. Then $x_0' = H'/E$ and $z'_1 = x_0 + z_1 - x_0'$,

and we wish to find z_2', the reinforcements to send.) It should be pointed out that the following analysis could apply to reinforcements arriving at any time after T_A, not just $T_A + \tau$. Graphically we have Figure 18 where T_{CR} is the time of control if reinforcements are sent and T_C is the time of control if no reinforcements are sent.

QUESTION: How many more men, z_2^*, should be dispatched at T_A?

From T_A on, we have $dy/dt = G_A - Ez_1t$ and $dy/dt = 0$ when $t = T_C = G_A/Ez_1$. Since reinforcements could not arrive until $T_A + \tau$, if $\tau > G_A/Ez_1$ send no reinforcements as the fire will be under control before they get there. If $\tau < G_A/Ez_1$ we should send reinforcements in order to control the fire at T_{CR}. Now, how much area will burn between T_A and T_{CR}? From T_A to $T_A + \tau$, we get

$$\int_{T_A}^{T_A+\tau} (G_A - Ez_1t)dt = \int_0^{\tau} (G_A - Ez_1t)dt$$

and from $T_A + \tau$ to T_{CR}, we get

$$\int_0^{\tau_{CR}} \left[G_R - E(z_1 + z_2)t\right] dt$$

where G_R = growth rate at $T_A + \tau = G_A - Ez_1\tau$ = rate when reinforcements arrive. Now, from $T_A + \tau$ to T_{CR}

$$\frac{dy}{dt} = G_R - E(z_1 + z_2)t$$

and when $dy/dt = 0$

$$t = \tau_{CR} = \frac{G_R}{E(z_1 + z_2)}$$

$\therefore$ the area burned from T_A to $T_{CR} = R = G_A\tau - Ez_1\tau^2/2 + G_R^2/2E(z_1 + z_2)$.

Now, how many man hours are spent fighting the fire? There are $x_0 + z_1$ men from T_A to T_{CR} and z_2 men from $T_A + \tau$ to T_{CR} .

$$\therefore \quad \text{total manhours} = S = (x_0 + z_1)(\tau + \tau_{CR}) + z_2\tau_{CR}$$

$$= (x_0 + z_1 + z_2)\frac{G_R}{E(z_1 + z_2)} + (x_0 + z_1)\tau$$

$$\therefore \quad C = C_B y_A + C_B R + C_F + C_S(x_0 + z_1 + z_2) + C_X S \quad .$$

To minimize this with respect to z_2 , we need only consider the following terms which include z_2 :

$$\frac{C_B G_R^2}{2E(z_1 + z_2)} + C_S z_2 + \frac{C_X(x_0 + z_1 + z_2)G_R}{E(z_1 + z_2)}$$

$$= \frac{C_B G_R^2}{2E(z_1 + z_2)} + C_S z_2 + \frac{C_X x_0 G_R}{E(z_1 + z_2)} + \frac{C_X z_1 G_R}{E(z_1 + z_2)} + \frac{C_X z_2 G_R}{E(z_1 + z_2)}$$

$$= \frac{C_B G_R^2}{2E(z_1 + z_2)} + C_S z_2 + \frac{C_X HG_R}{E^2(z_1 + z_2)} + \frac{C_X G_R}{E}$$

$$\frac{\partial C}{\partial z_2} = C_S - \frac{C_B G_R^2}{2E(z_1 + z_2)^2} - \frac{C_X HG_R}{E^2(z_1 + z_2)^2} = 0$$

$$(z_1 + z_2^*)^2 = \frac{EC_B G_R^2 + 2C_X H G_R}{2E^2 C_S}$$

$$\therefore \quad z_2^* = \sqrt{\frac{2C_X H G_R + EC_B G_R^2}{2E^2 C_S}} - z_1$$

and

$$C^* = C_B y_A + C_B V + C_F + C_S(x_0 + z_1 + z_2^*) + C_X W$$

where

$$V = G_A \tau - \frac{E z_1 \tau^2}{2} + \frac{G_R^2}{2E(z_1 + z_2^*)}$$

and

$$W = (x_0 + z_1 + z_2^*) \frac{G_R}{E(z_1 + z_2^*)} + (x_o + z_1)\tau$$

Now, what is the cost if no reinforcements are sent? From (18)

$$C = C_B y_A + C_B \frac{G_A^2}{2Ez_1} + C_F + C_S(x_0 + z_1) + C_X \frac{G_A(x_0 + z_1)}{Ez_1}$$

$\therefore$ savings by sending in reinforcements, z_2^* is

$$C - C^* = C_B \left(\frac{G_A^2}{2Ez_1} - V \right) - C_S z_2^* + C_X \left[\frac{G_A(x_0 + z_1)}{Ez_1} - W \right]$$

where V and W are given above.

VIII. Possible Extensions and Modifications

There are undoubtedly numerous extensions and modifications to this model structure that would give further insight into the initial attack problem. Some of the more important would probably include:

1) Investigating the relationship between C_S and τ_A. Since there are only a limited number of transportation possibilities, i.e., foot, truck, plane, helicopter, etc., τ_A will be directly affected by the method used.

2) Examining in detail the effect of reducing τ_A on C^* and z^*, We see that reducing τ_A will reduce the number of suppression forces and perhaps the cost, but doing this will probably increase C_S, and after a certain point the cost will actually increase. This point needs further study.

3) Moving backward in the model to look at τ_D, the time from ignition to detection to see what the possibilities are of reducing this, and then examining the effects on C^* and z^*.

4) Examining the decisions involved in selecting between two or more strategies with different values of certain parameters.

5) Developing realistic values for the various parameters under varying conditions to simulate actual fire conditions encountered in the field.

6) Incorporating nonlinear cost elements into the cost structure.

7) Investigating the functional relationships between E, G_A, and y_A.

8) Relating the model to fire-fighting tactical considerations.

IX. Conclusions – The Model

In this report we have presented a simple model which gives an indication of the size of suppression forces which will be most economical for fighting a given fire. Naturally, this model should be accepted only if the assumptions presented herein are <u>substantially</u> correct.

However, even if not all features of a given fire are included herein, it should not be concluded that the results obtained are worthless. First of all, a simple model will provide a first approxiamtion to reality – to which later refinements will only result in small changes. Secondly, when simple models are elaborated, it usually happens that the same cluster of parameters and functional dependecies remain substantially the same; thus if experience indicates, for instance, that the results here obtained are 10% too small – this should not be surprising. But, it is not likely that a model should suggest a decrease in forces when experience suggests an increase.

Finally, a simple model provides at least a framework in which various technological problems can be tentatively answered, and various proposals ranked for further consideration.

Further investigation and testing of the hypothesis of this model will, we hope, provide a realistic basis for the ultimate use of this model in decision making. This report is only the first chapter.

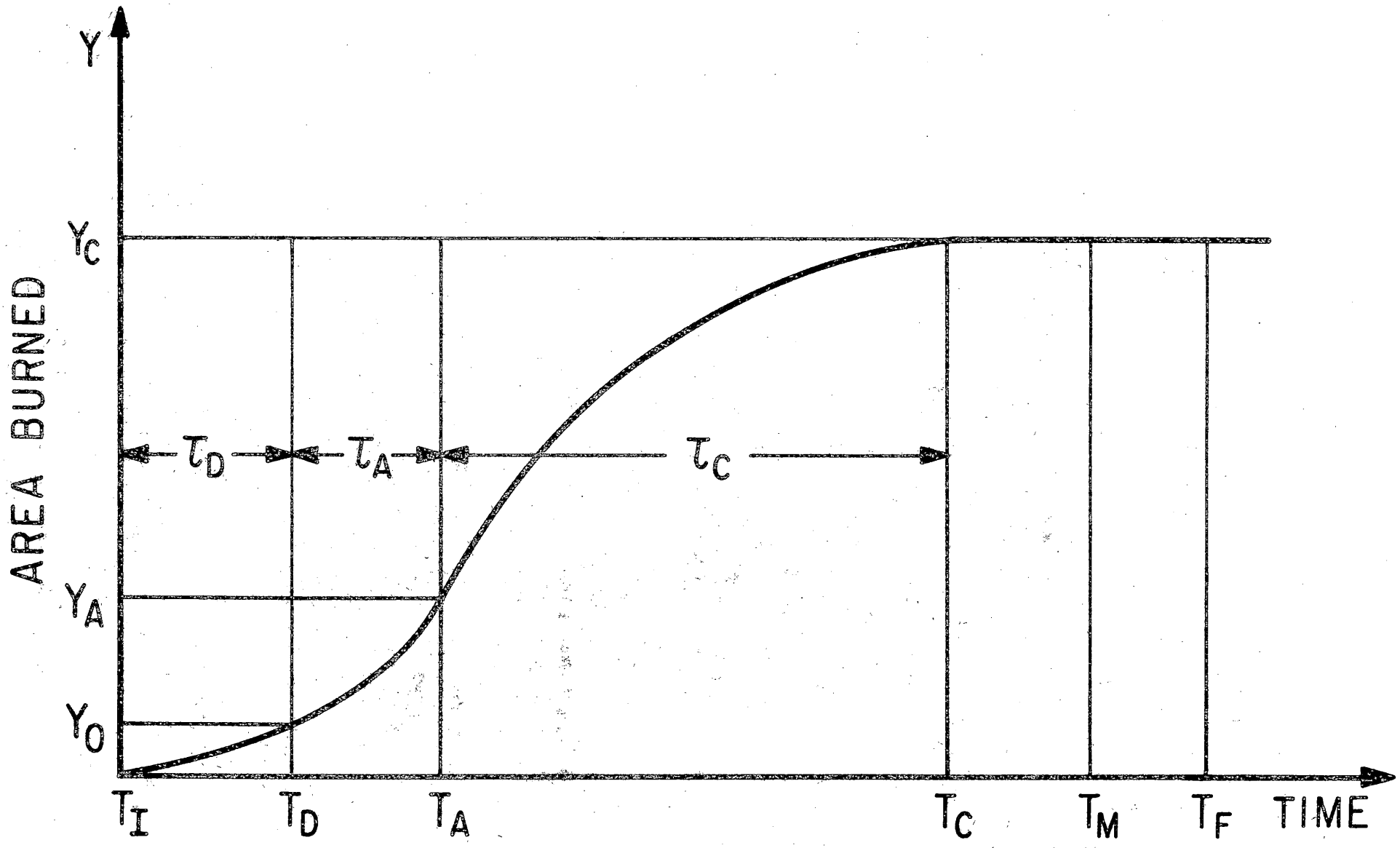

FIG. I PATTERN OF WILDLAND FIRE GROWTH

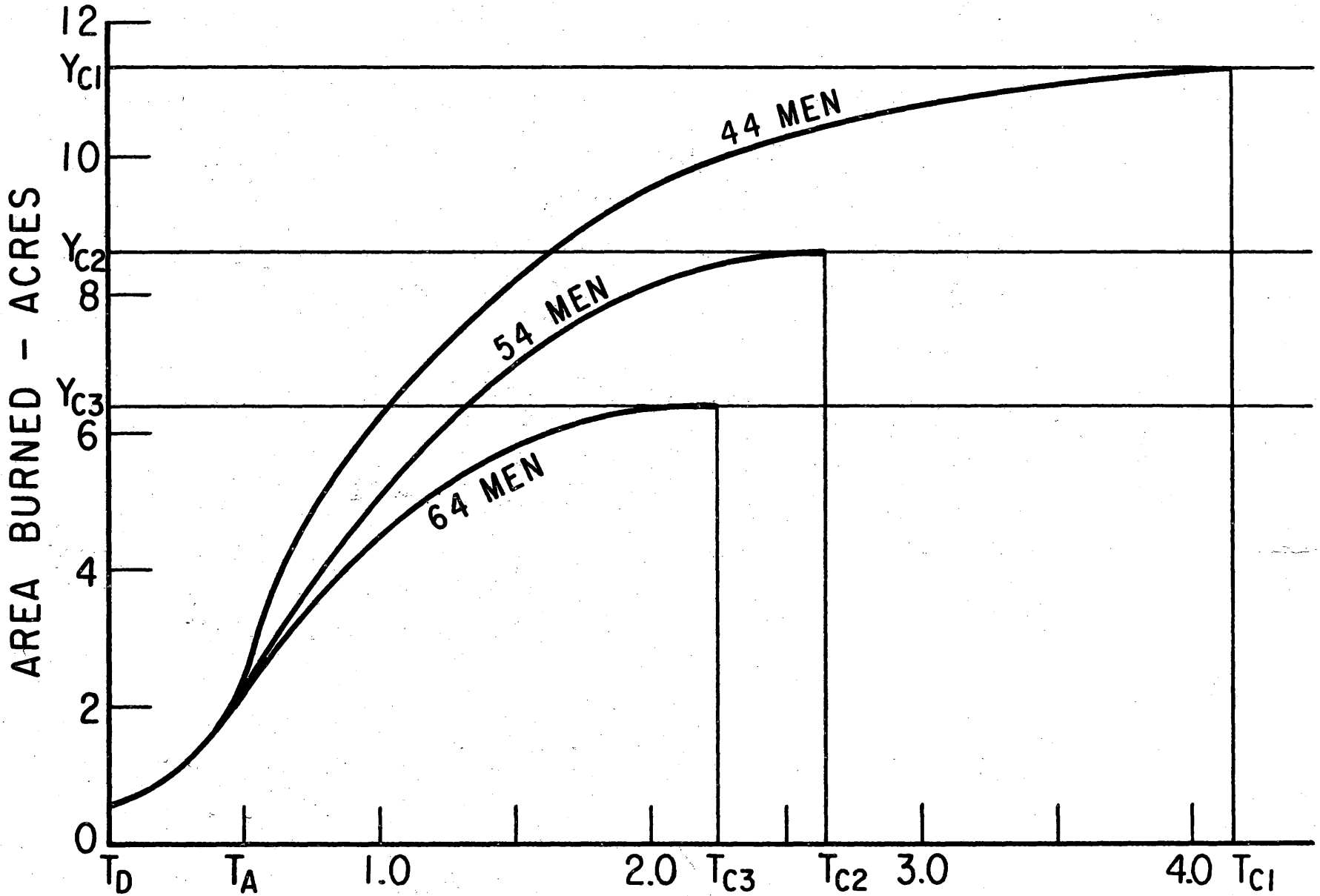
12
Y_{C1}
10
Y_{C2}
8
Y_{C3}
6
4
2
0
AREA BURNED - ACRES
44 MEN
54 MEN
64 MEN
T_D
T_A
1.0
2.0
T_{C3}
T_{C2}
3.0
4.0
T_{C1}
TIME - HOURS

FIG. 2

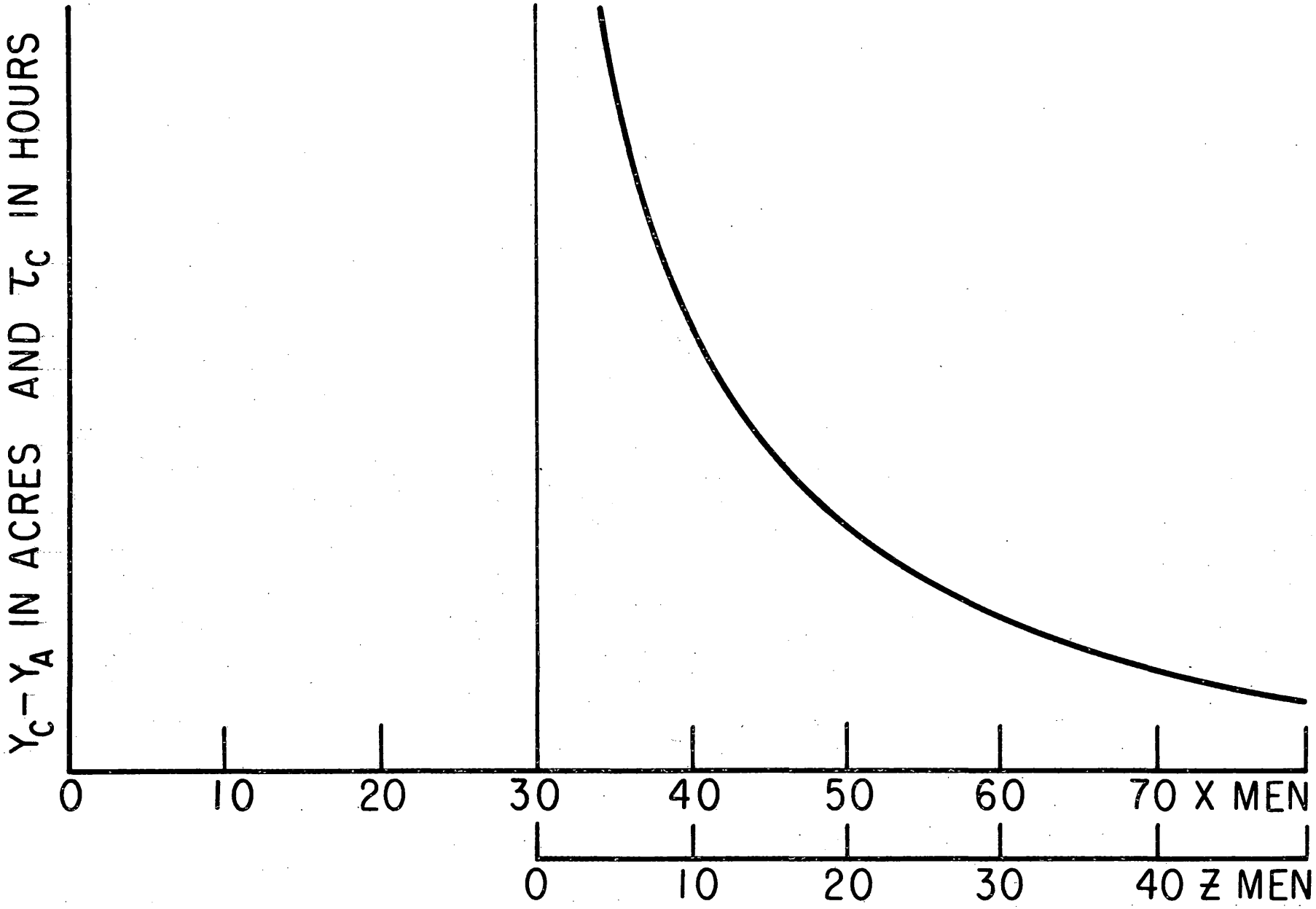
$Y_C - Y_A$ IN ACRES AND τ_C IN HOURS
0
10
20
30
40
50
60
70 X MEN
0
10
20
30
40 Z MEN

FIG. 3

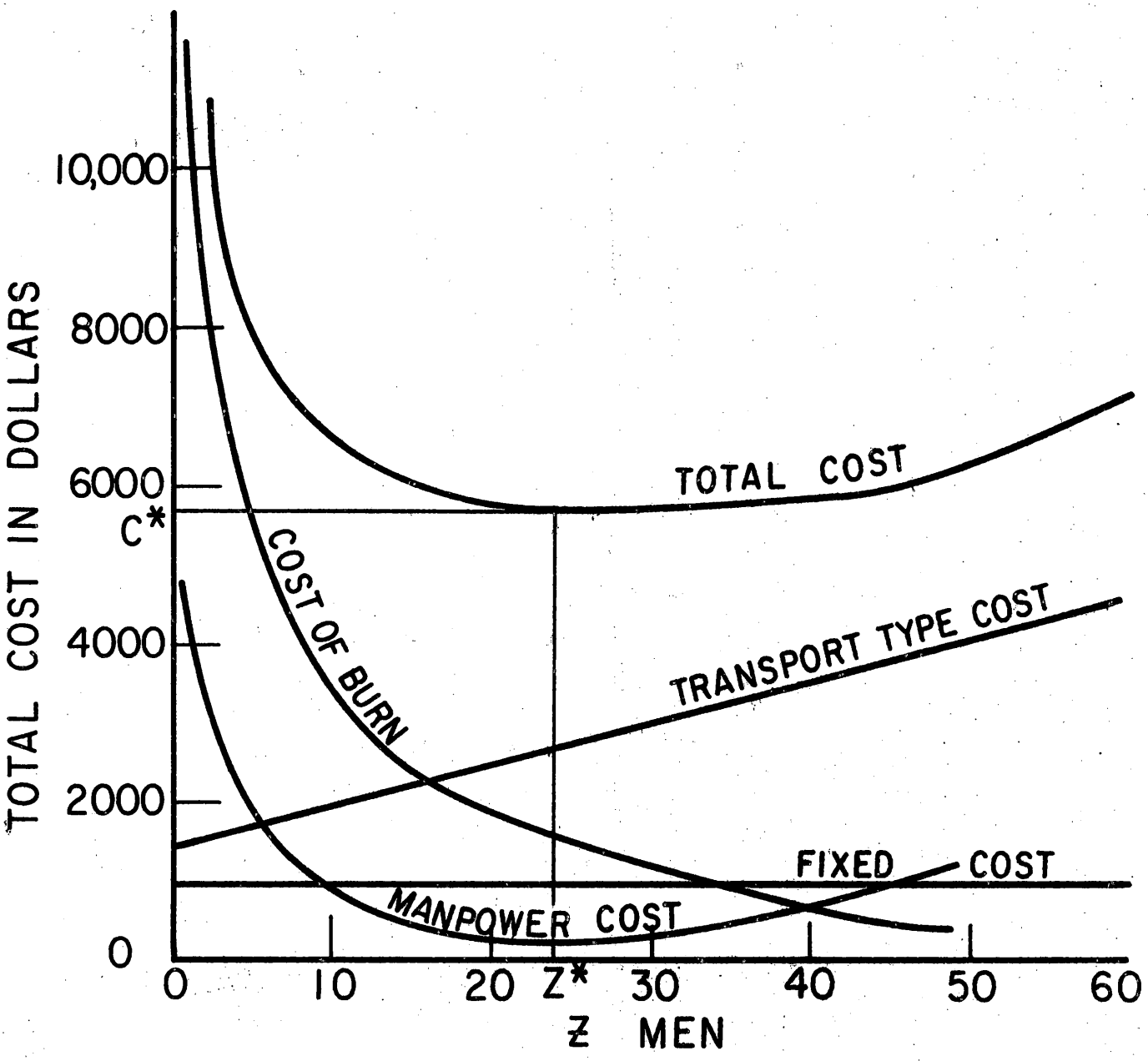

FIG. 4 TOTAL COST AND ITS COMPONENTS

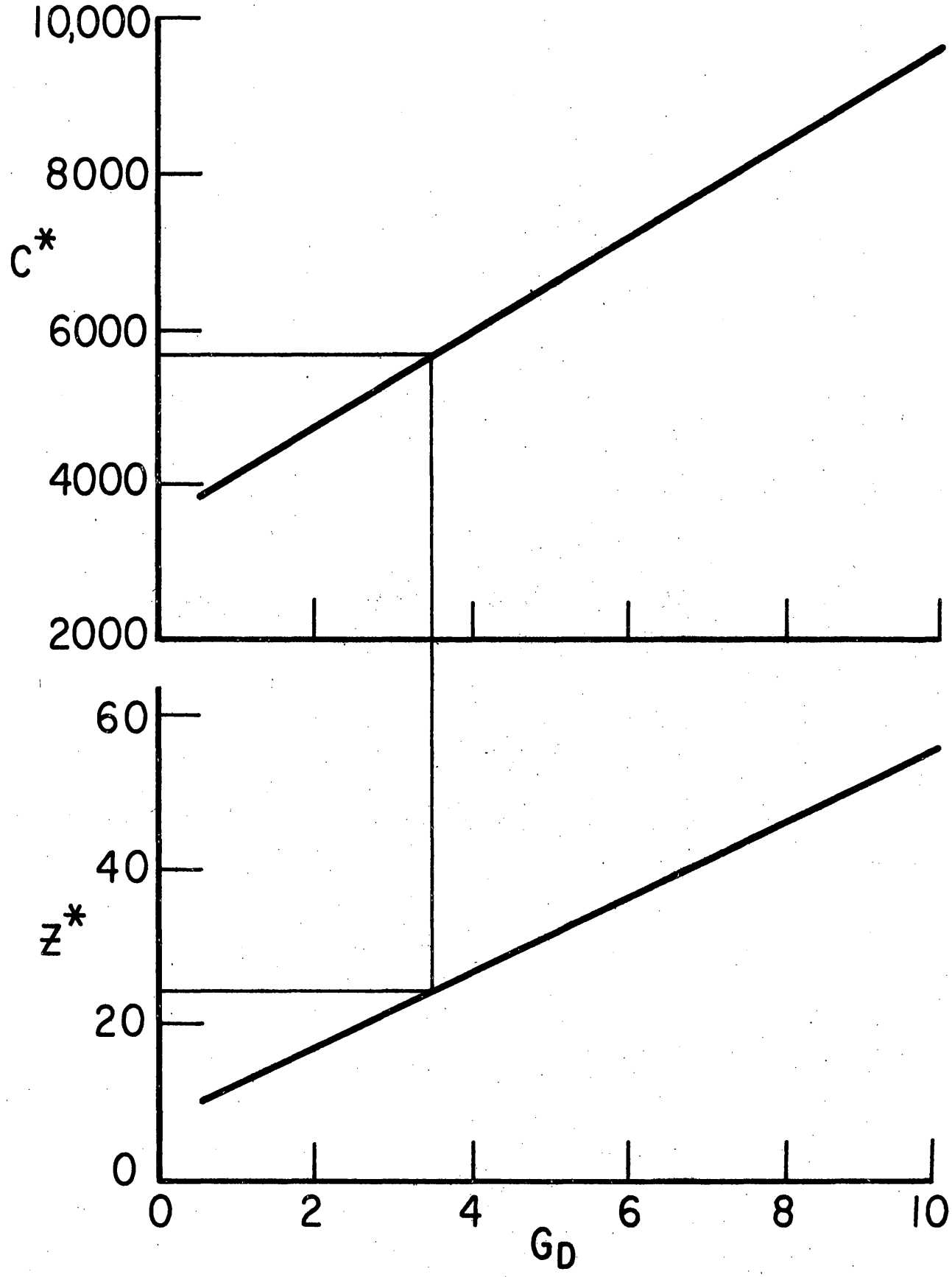

FIG. 5 and 6

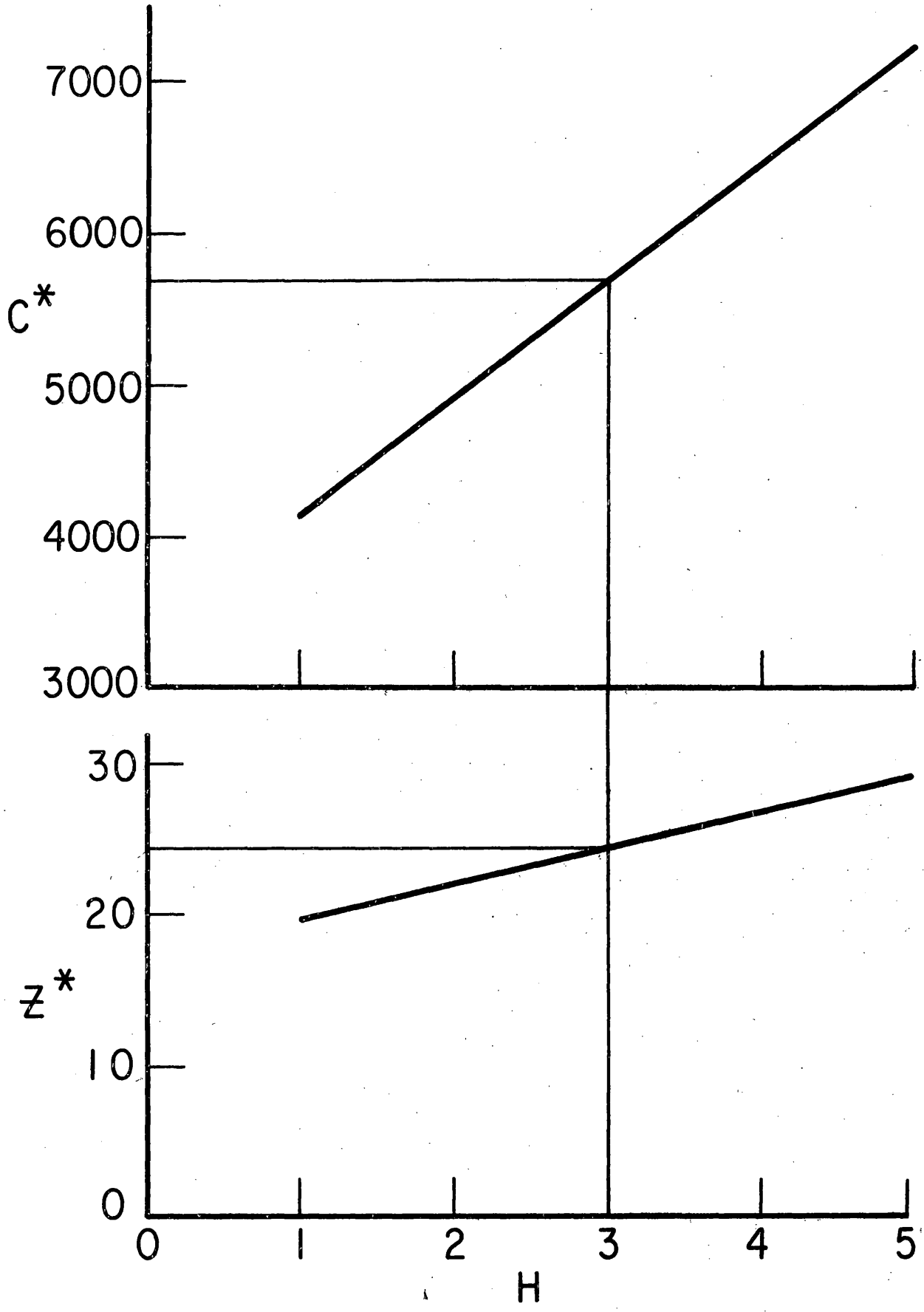

FIG. 7 and 8

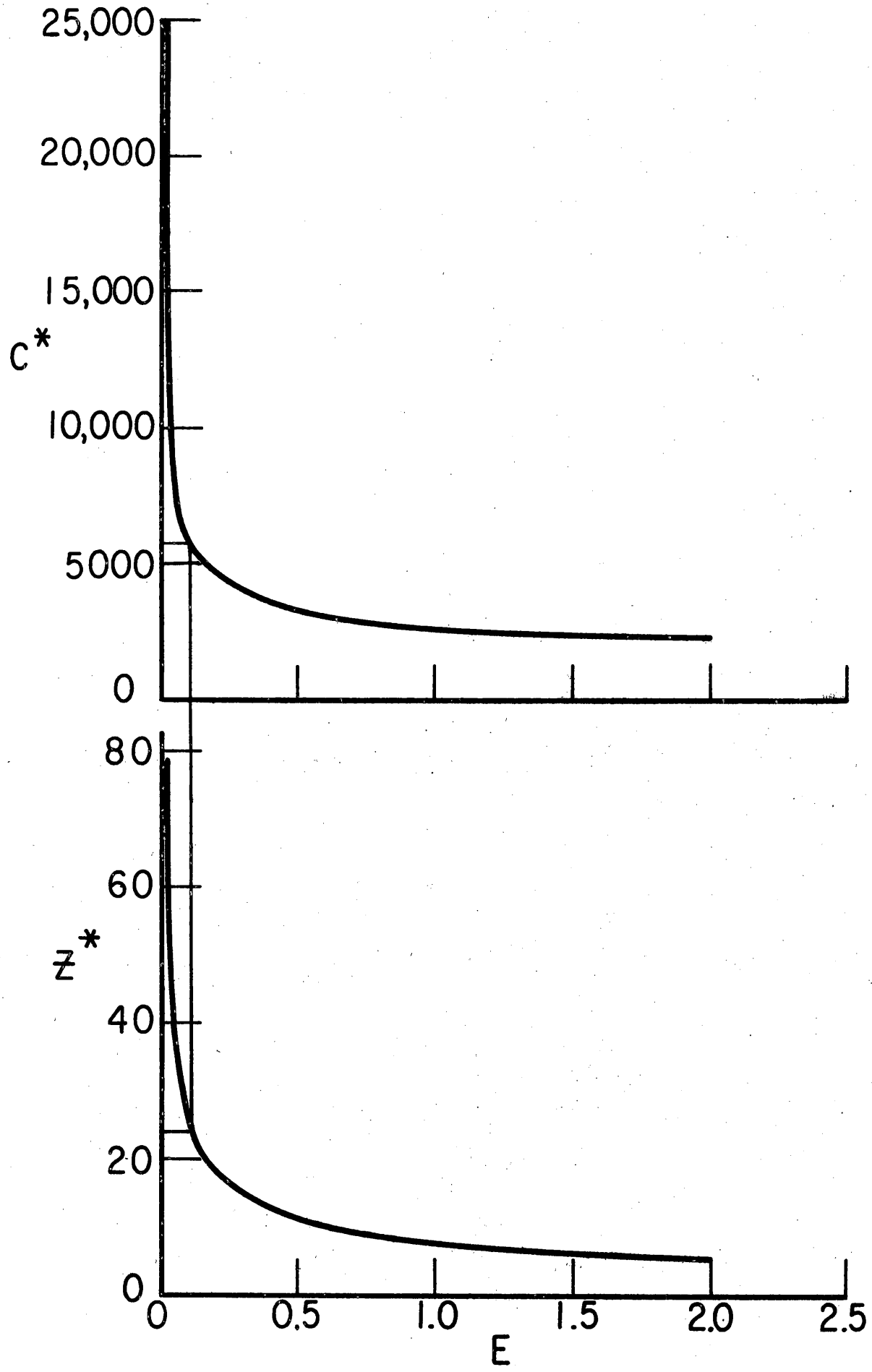

FIG. 9 and 10

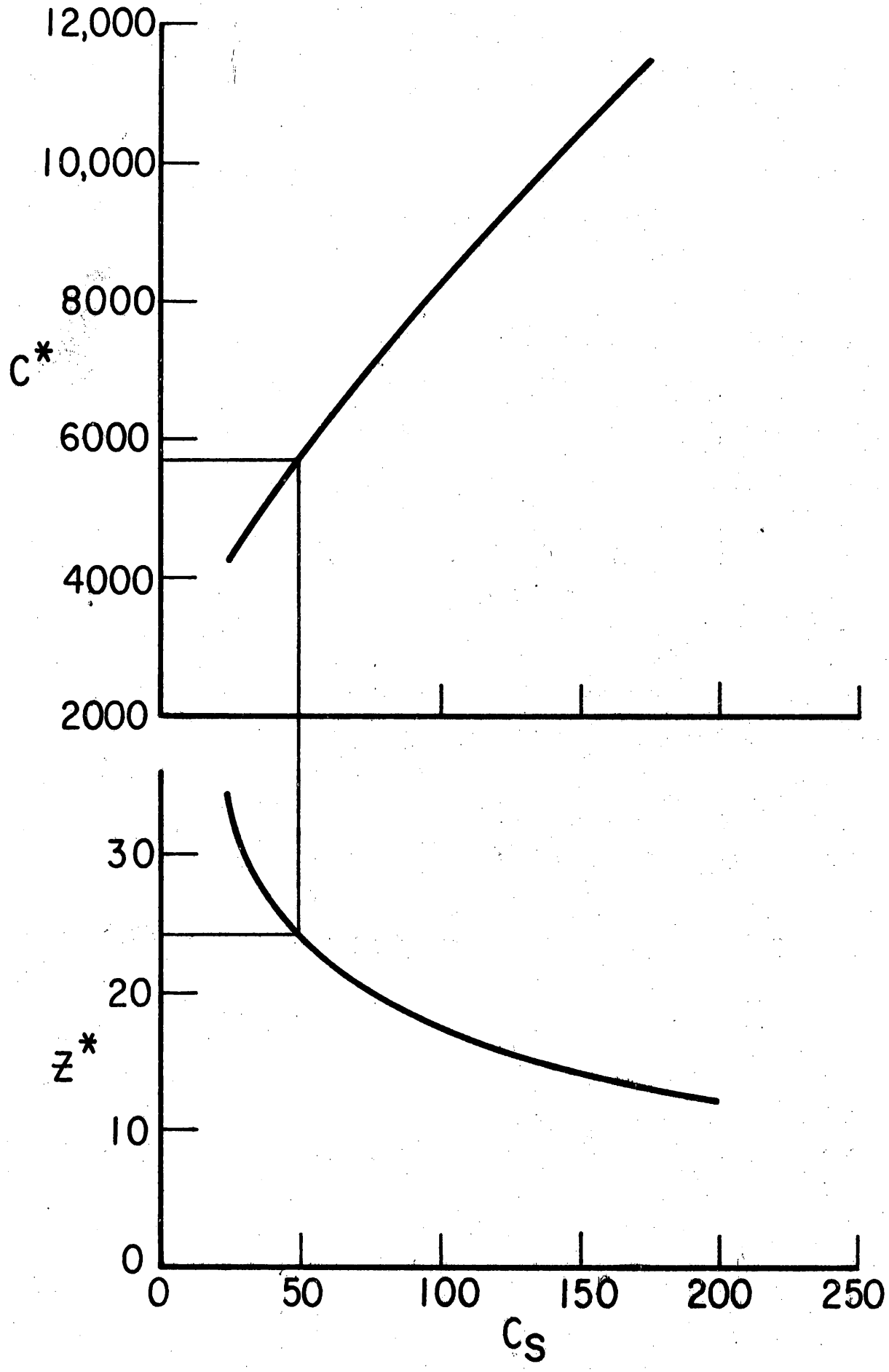
12,000
10,000
8000
C*
6000
4000
2000
30
20
Z*
10
0
0
50
100
150
200
250
Cs

FIG. 11 and 12

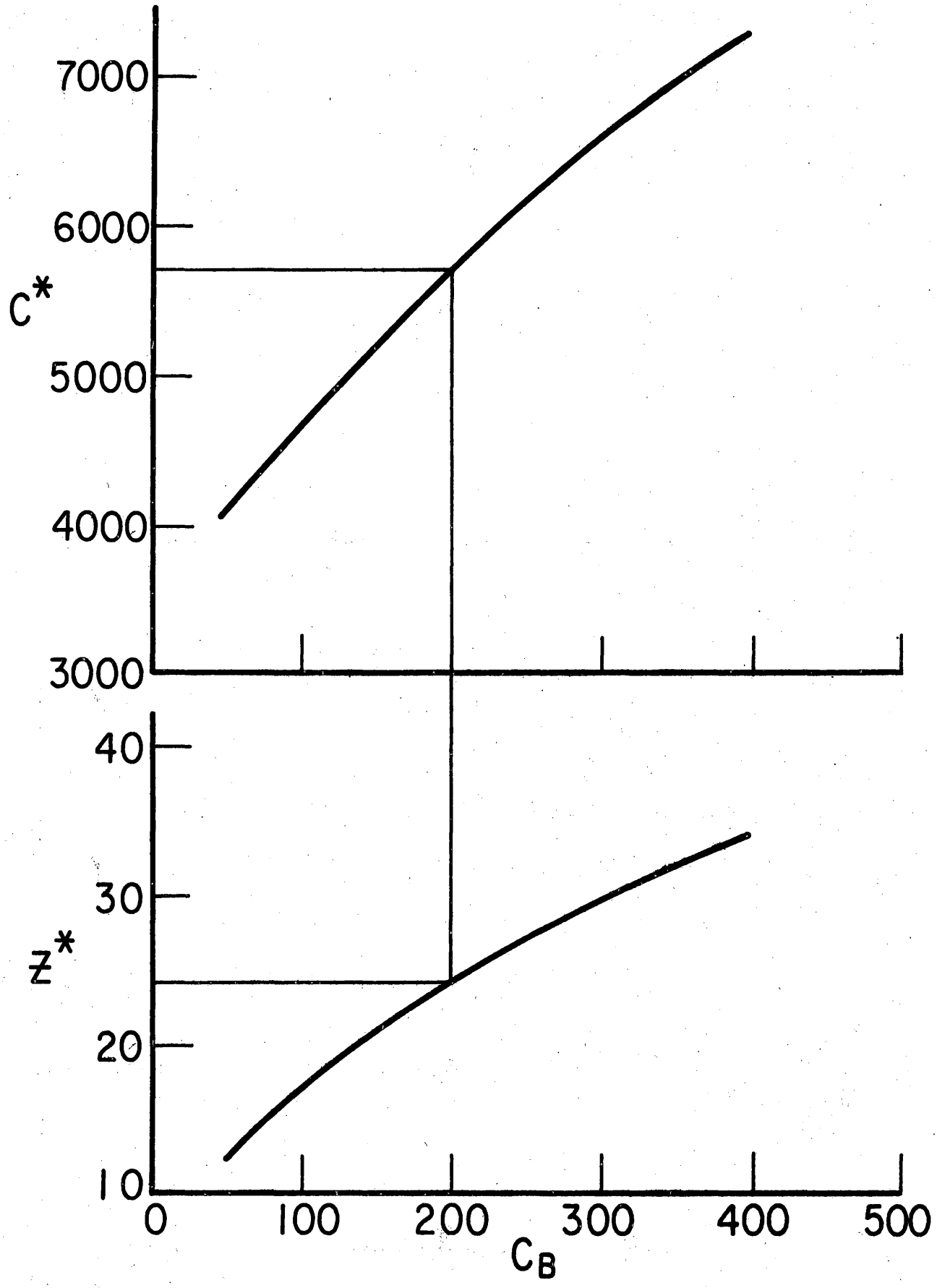

FIG. 13 and 14

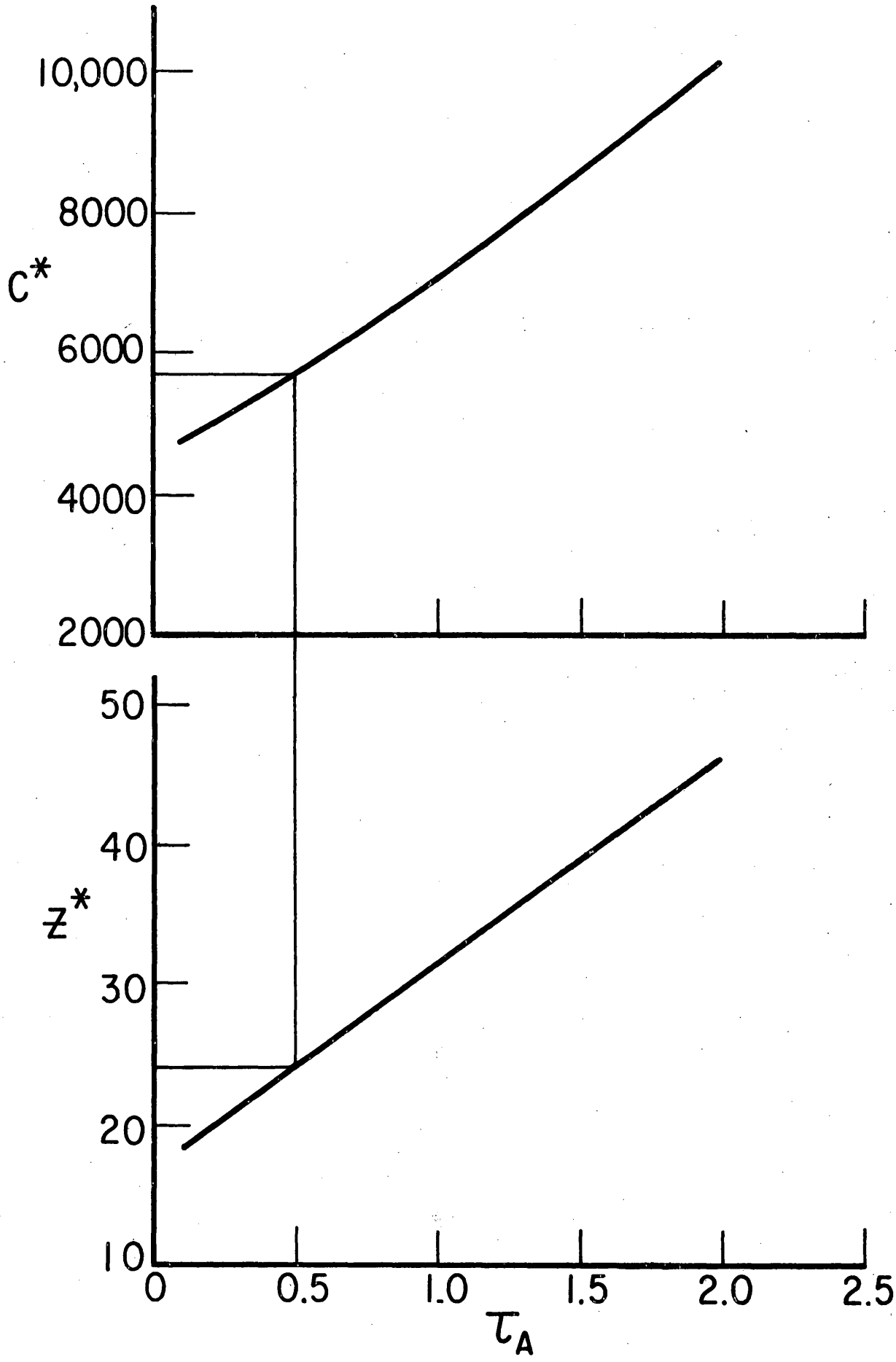

FIG. 15 and 16

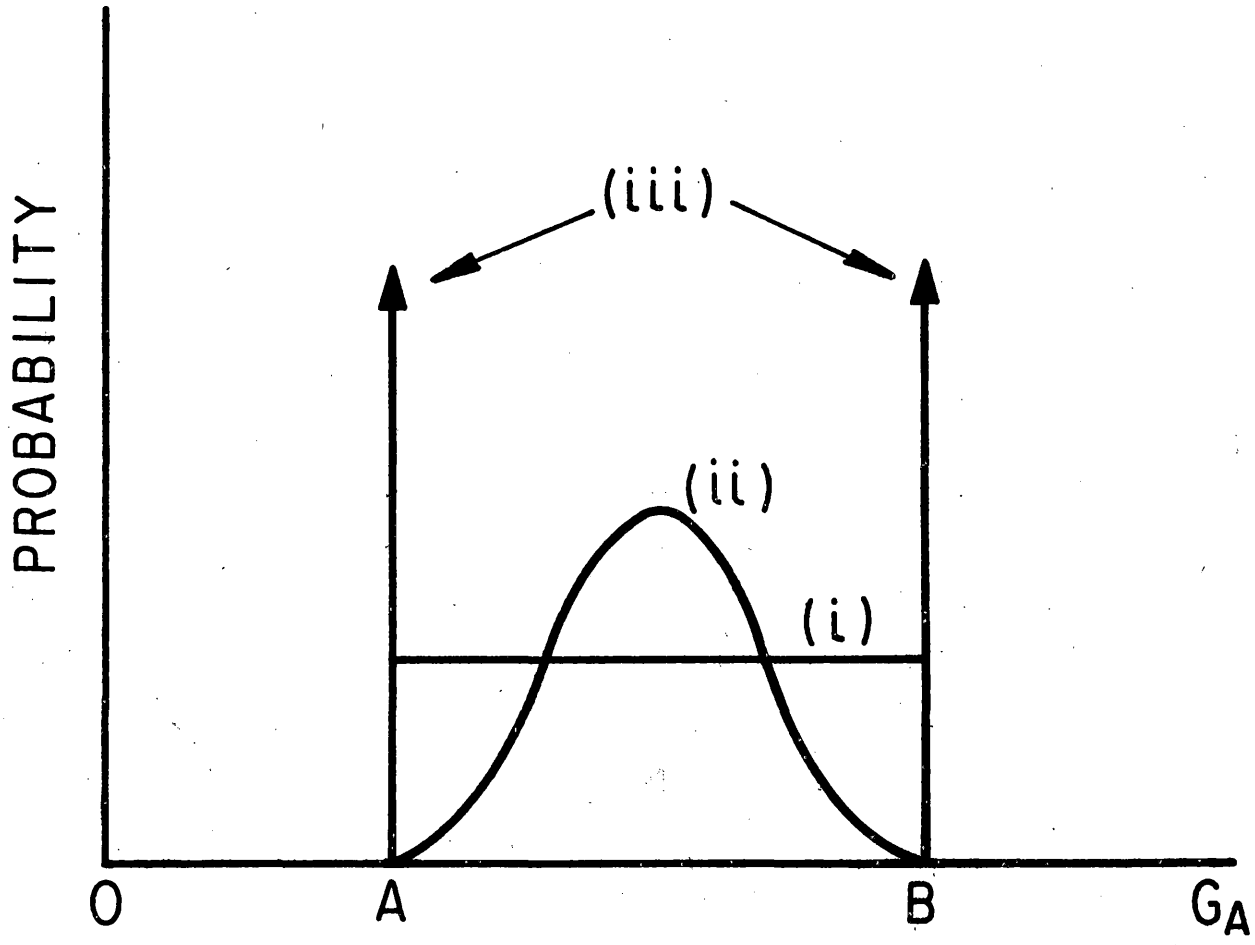

FIG. 17 POSSIBLE DISTRIBUTIONS OF GROWTH RATE

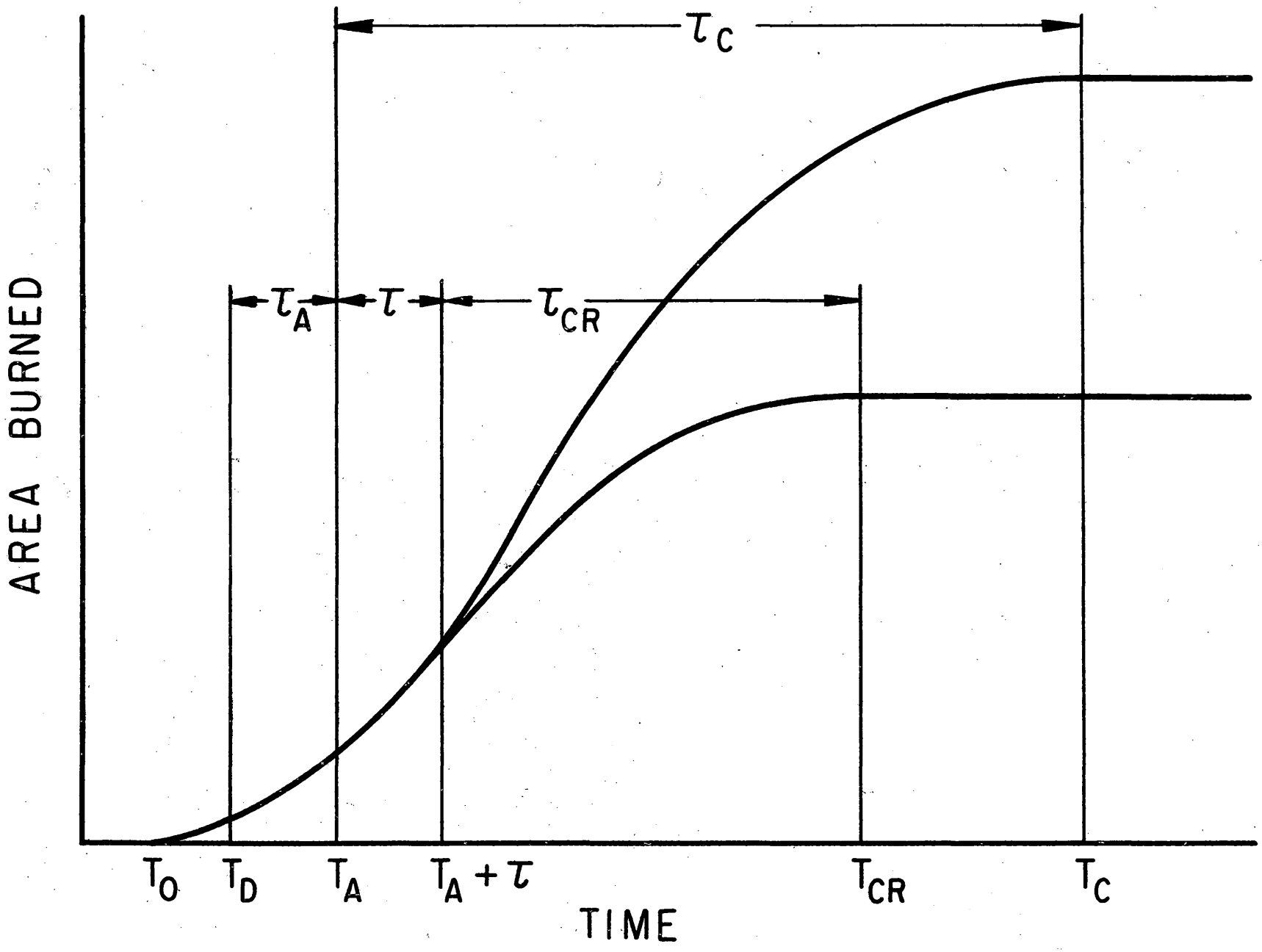

FIG. 18